DIBARYON CONDENSATE IN NUCLEAR MATTER AND NEUTRON STARS: EXACT ANALYSIS IN ONE-DIMENSIONAL MODELS

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Abstract

Phase transitions of nuclear matter to the quark-gluon plasma with subsequent restoration of chiral symmetry have been widely discussed in the literature. We investigate the possibility for occurrence of dense nuclear matter with a dibaryon Bose-Einstein condensate as an intermediate state below the quark-gluon phase transition. An exact analysis of this state of matter is presented in a one-dimensional model. The analysis is based on a reduction of the quantization rules for the N-body problem to N coupled algebraic transcendental equations. We observe that when the Fermi momentum approaches the resonance momentum, the one-particle distribution function increases near the Fermi surface. When the Fermi momentum is increased beyond the resonance momentum, the equation of state becomes softer. The observed behavior can be interpreted in terms of formation of a Bose-Einstein condensate of two-fermion resonances (e.g. dibaryons). In cold nuclear matter, it should occur at $2(m_N+\varepsilon_F) \geq m_D$ where m_N and m_D are the nucleon and dibaryon masses and ε_F is the nucleon Fermi energy.

1 INTRODUCTION

Exactly solvable models are of considerable interest. They provide important tests of different approximation schemes. These approximation methods may then be

applied with increased confidence to more complex cases where analytic solutions are impossible. In fact, one-dimensional models very often provide the only possibility to gain analytic insights into the behavior of an interacting many-body system that go beyond perturbation theory. In this sense, exactly solvable low-dimensional models may provide valuable guidance in finding the proper dynamical description of more complicated systems.

The recent discovery [1] of Bose-Einstein condensation in a dilute vapor of rubidium-87 atoms created new interest in the phenomenon of Bose-Einstein condensation and in exactly solvable models of interacting bosons. Already some time ago [2] it was noted that the energy spectrum of a one-dimensional Bose-gas of nonpenetrable particles is identical to that of an ideal Fermi gas. In other words, there is an exact one-to-one correspondence between bosons which interact via an "infinite wall" two-body potential, and a system of noninteracting fermions. Later [3, 4] an exact analytic solution for a one-dimensional Bose-gas interacting via a delta-function potential $V(x) = \alpha\delta(x)$, where α is an arbitrary positive parameter was obtained. It was shown that the model of ref. [2] is a special case of the model in refs. [3, 4], since in the limit $\alpha \to \infty$ particles do not penetrate each other. For $\alpha < \infty$ the qualitative features of the energy spectrum remain unaltered. A class of exact solutions three particles with different masses interacting through a finite-strength delta-function potential was found [5]. Thermodynamic properties of an interacting Bose-gas with the potential $V(x) = \alpha\delta(x)$ were also discussed [6]. It was demonstrated that at finite temperatures the thermodynamic functions do not show a non-analytic behavior. In ref. [7] the possibility of superfluidity at zero temperature is discussed. In ref. [8] it was shown that long-range correlations inherent in the models of refs. [2, 3] decrease as some power of $1/x$ at $T = 0$. One-dimensional models have also led to a deeper understanding of critical phenomena [9]. Exact results for the quantization of the Toda lattice are reviewed in ref. [10].

There is an extensive literature devoted to dibaryon resonances. Dibaryons are predicted by most low-energy models of quantum chromodynamics (QCD) [11-14]. Some candidates are reported to be seen in experiments [15, 16]. The appearance of Bose-Einstein condensation of two-fermion resonances in nuclear matter softens the equation of state of nuclear matter and decreases the upper limits for masses of neutron stars [17]. In the framework of the Walecka model, heterogenous nucleon-dibaryon matter is discussed in ref. [18]. In the present paper, we focus on one-dimensional Fermi-systems with a Bose-resonance in the two-fermion channel. These one-dimensional models might have important implications for the behavior of dibaryon resonances in the nuclear medium.

Previous results [2-8] were obtained for a δ-function interaction potential. Here,

we show that one-dimensional models admit exact solutions for other types of zero-range singular potentials. We classify eigenfunctions of the N-body Hamiltonian, establish quantization rules, find eigenvalues of the N-body Hamiltonian without recourse to perturbation theory. In the thermodynamic limit one can exactly (numerically) calculate dispersion laws for elementary excitations, the ground state energy, and the equation of state.

The outline of the paper is as follows. We start with discussing the simpler case of a system of bosons interacting through an arbitrary finite-range potential $V(x)$ ($V(x) \neq 0$ for $\mid x \mid \leq a$ and $V(x) = 0$ for $\mid x \mid > a$) generating a nontrivial two-body scattering S-matrix. We then pass to the limit $V(x) \to \infty$ for $\mid x \mid \leq a$ and $a \to 0$. In this limit, there is a wide class of nontrivial finite two-body S-matrices. In sect. 2, we establish the properties of the coefficients entering the plane wave expansion of the N-boson wave function. In particular, we study their properties with respect to permutations of the particle quasi-momenta and establish relations between these coefficients. Then we discuss the symmetry of the wave function under permutations of the arguments, periodic boundary conditions, and matching conditions for the wave function. In Sect. 3, we derive the generalized Bohr-Sommerfeld quantization rules which reduce the quantization problem to the problem of finding solutions to N coupled algebraic transcendental equations. In the limit $a \to 0$, the energy eigenvalues are completely determined by the two-particle scattering phase shifts. In Sect. 4, we discuss the thermodynamic limit $N \to \infty$, $L \to \infty$, $N/L = constant$. A linear integral equation is derived for the distribution function of the quasi-momenta of the particles. The distribution function is completely determined by the scattering phase shifts of the particles. The elementary excitations in the Bose-system are classified and their dispersion laws are established. In Sect. 5, the results of Sects. 2 and 3 are extended to special classes of exactly solvable Fermi-systems. In Sect. 6, we discuss the properties of a system of fermions interacting via potentials which allow for a resonance in the two-body S-matrix. In Sect. 7 the problem of stability of self-gravitating objects (neutron stars) made up of fermions with a resonance interaction is discussed. In one dimension all objects of such a kind are stable in general. In three-dimensional space narrow resonances cause instability of massive neutron stars. Finally, in sect. 8 we summarize and discuss the results.

2 EXPANSION COEFFICIENTS OF THE WAVE FUNCTION

We search for solutions of the N-body Schrödinger equation in one dimension

$$(\sum_{j}^{N}(\frac{\hat{p}_j^2}{2m}+\sum_{i<j}V(x_i-x_j))\Psi(x_1,...,x_N)=E\Psi(x_1,...,x_N) \tag{2.1}$$

in the interval $[0, L]$. Here, the particles with mass m are assumed to be bosons, so the wave function $\Psi(x_1, ..., x_N)$ is symmetric under permutation of any pair of its arguments. The potential $V(x_i - x_j)$ is assumed to vanish for $|x_i - x_j| > a$. Finally, we pass to the limit $V(x) \to \infty$ for $\mid x \mid < a$ and $a \to 0$. In this limit there exists a wide class of nontrivial two-body S-matrices.

The problem considered here is quite similar to the one-dimensional Heisenberg model of ferromagnetism with nearest neighbor interactions, first solved by Bethe [19].

Following refs. [2, 3], we impose periodic boundary conditions for the wave function

$$\Psi(x_1,...,x_j=0,...,x_N)=\Psi(x_1,...,x_j=L,...,x_N) \tag{2.2}$$

for any j.

One can verify that the function

$$\chi(x_1,...,x_N)=\exp(i\sum_{j}^{N}k_jx_j) \tag{2.3}$$

satisfies the Schrödinger equation, if $x_j + a < x_{j+1}$, i.e. when $V(x_i - x_j) = 0$ for any pair of the arguments. The energy equals

$$E=\sum_{j}^{N}\frac{k_j^2}{2m}. \tag{2.4}$$

It is evident that any function which differs from Eq. (2.3) by a permutation of the particle quasi-momenta $(k_1, ..., k_N) \to (k_{\alpha_1}, ..., k_{\alpha_N})$ satisfies the same equation and has the same energy. The problem reduces therefore to (i) the determination of the weights of all components $(k_{\alpha_1}, ..., k_{\alpha_N})$ of the wave function $\Psi(x_1, ..., x_N)$, (ii) matching the wave functions in the different regions of integrability ("A-regions") where the exact wave function can be represented as a superposition of plane waves (2.3), (iii) symmetrization of the expression (2.3) with respect to the arguments, (iv) taking into account periodic boundary conditions, and finally (v) derivation of the multidimensional analog of the Bohr-Sommerfeld quantization rules for the particle quasi-momenta k_i.

Let us order the arguments of the wave function in increasing sequence, e.g. $x_5 < x_1 < x_3 < ... < x_{12}$. The numbers $(5, 1, 3, ..., 12)$ constitute a set $(\gamma_1, ..., \gamma_N)$. Therefore, γ_i is the number of the argument that occupies the i-th place in the above ordered sequence. Each A-region can be brought into correspondence with a set $(\gamma_1, ..., \gamma_N)$ which is a permutation of the numbers $(1, ..., N)$. There are $N!$ regions in which solutions can be represented in form of plane waves. Each A-region is fixed by a set of inequalities

$$x_{\gamma_j} + a < x_{\gamma_{j+1}} \tag{2.5}$$

for $j = 1, N-1$, so that $V(x_i - x_j) = 0$ for any pair of the arguments. The solutions of the Schrödinger equation in the region $(\gamma_1, ..., \gamma_N)$ can be written in the form

$$\Psi(x_1, ..., x_N) = \sum_{\alpha_1 ... \alpha_N} C^{\gamma_1 ... \gamma_N}_{\alpha_1 ... \alpha_N} \exp(i \sum_j^N k_j x_j). \tag{2.6}$$

The sum is taken over the $N!$ permutations of the $(k_1, ..., k_N)$. It is assumed that the k's are all different. It will be shown below that this is a general case.

The wave function $\Psi(x_1, ..., x_N)$ contains for finite-range potentials terms with different sets of the particle quasi-momenta $\{k_i\}$. In the limit $a \to 0$, however, only one unique set $\{k_i\}$ survives. The requirement $a \to 0$ is necessary to ensure completeness of the plane wave expansion in the A-regions.

Now, we consider the periodic boundary conditions. Suppose the set of arguments of the wave function on the left hand side of Eq. (2.2) belongs to the region $(\gamma_1, ..., \gamma_N)$, then $\gamma_1 = j$ due to the condition $x_j = 0$. The set of arguments of the wave function on the right hand side of the Eq. (2.2) belongs to the region $(\lambda_1, ..., \lambda_N)$.

It is clear that $\lambda_i = \gamma_i + 1$ for $i < N$, $\lambda_N = \gamma_1$ by virtue of $x_j = L$. The terms of the sum (2.6) are all linearly independent. Therefore, the periodic boundary condition (2.2) can be unambiguously projected to the expansion coefficients

$$C^{\gamma_1 ... \gamma_{N-1}, \gamma_N}_{\alpha_1 ... \alpha_{N-1}, \alpha_N} = C^{\gamma_2 ... \gamma_N, \gamma_1}_{\alpha_2 ... \alpha_N, \alpha_1} \exp(i k_{\alpha_1} L). \tag{2.7}$$

We illustrate this with an example. Let $N = 2$, $j = 1$ then

$$\Psi(0, x_2) = C^{12}_{12} \exp(i k_2 x_2) + C^{12}_{21} \exp(i k_1 x_2),$$
$$\Psi(L, x_2) = C^{21}_{12} \exp(i k_1 x_2 + i k_2 L) + C^{21}_{21} \exp(i k_2 x_2 + i k_1 L).$$

From the condition $\Psi(0, x_2) = \Psi(L, x_2)$ follows $C^{12}_{12} = C^{21}_{21} \exp(i k_1 L)$ and $C^{12}_{21} = C^{21}_{12} \exp(i k_2 L)$.

Under exchange of the coordinates $x_{\gamma_s} \leftrightarrow x_{\gamma r}$ one region of integrability transforms to another one. The symmetry conditions for the wave functions under these

permutations can be formulated in terms of the expansion coefficients in the following way

$$C^{\gamma_1\dots\gamma_s\dots\gamma_r\dots\gamma_N}_{\alpha_1\dots\alpha_s\dots\alpha_r\dots\alpha_N} = C^{\gamma_1\dots\gamma_r\dots\gamma_s\dots\gamma_N}_{\alpha_1\dots\alpha_s\dots\alpha_r\dots\alpha_N}. \tag{2.8}$$

Let us illustrate this again with an example. For $N = 2$, $x_1 < x_2$ we get

$$\Psi(x_1, x_2) = C^{12}_{12} \exp(ik_1x_1 + ik_2x_2) + C^{12}_{21} \exp(ik_2x_1 + ik_1x_2),$$
$$\Psi(x_2, x_1) = C^{21}_{12} \exp(ik_1x_1 + ik_2x_2) + C^{21}_{21} \exp(ik_2x_1 + ik_1x_2).$$

The condition $\Psi(x_1, x_2) = \Psi(x_2, x_1)$ implies $C^{12}_{12} = C^{21}_{12}$, $C^{12}_{21} = C^{21}_{21}$.

We consider the matching conditions for the wave functions in different A-regions. Consider first the two regions $\Gamma_\pm$ which can be obtained from each other by a permutation of the arguments x_{γ_s} and $x_{\gamma r}$ such that $s \pm 1 = r$. In the sequence of increasing arguments of the wave function, the values x_{γ_s} and $x_{\gamma r}$ occupy neighboring places. In the region Γ_+, x_{γ_s} is on the right, i.e. $s + 1 = r$, while in the region Γ_-, $x_{\gamma r}$ is on the left, i.e. $s - 1 = r$. Let $\xi_1 = x_{\gamma s}$, $\xi_2 = x_{\gamma r}$. We now join the regions Γ_+ and Γ_- and remove the restriction $|\xi_1 - \xi_2| > a$. In the region obtained (denoted herewith by Γ), we are looking for solutions of the Schrödinger equation in the form

$$\exp(i \sum_j^N k_j x_j)\chi(\xi_1, \xi_2). \tag{2.9}$$

The sum is extended over $j \neq s, r$. Let $q_1 = k_{\alpha s}$, $q_2 = k_{\alpha r}$. In the region Γ, the function $\chi(\xi_1, \xi_2)$ satisfies the equation

$$(\frac{\hat{p}_1^2}{2m} + \frac{\hat{p}_2^2}{2m} + V(\xi_1 - \xi_2))\chi(\xi_1, \xi_2) = E'\chi(\xi_1, \xi_2) \tag{2.10}$$

with $E' = \frac{q_1^2}{2m} + \frac{q_2^2}{2m}$. The interval $|\xi_1 - \xi_2| < a$, in which the potential is different from zero, is interesting only for supplying the correct matching conditions for the wave function for $\xi_2 - \xi_1 > a$ and $\xi_2 - \xi_1 < -a$. In terms of the total and relative quasi-momenta of the particles, $K = q_1 + q_2$, $k = (q_2 - q_1)/2$, the center-of-mass coordinate $X = (\xi_1 + \xi_2)/2$, and relative coordinate $x = \xi_2 - \xi_1$, the value $\chi(\xi_1, \xi_2)$ can be written for $x < -a$ in the form

$$\exp(iq_1\xi_1 + iq_2\xi_2) = \exp(iKX + ikx). \tag{2.11}$$

If the incoming plane wave $\exp(ikx)$ is part of the solution, the outgoing plane wave $\exp(-ikx)$ exists as well:

$$\exp(iKX - ikx) = \exp(iq_2\xi_1 + iq_1\xi_2). \tag{2.12}$$

These two waves differ from each other by permutation of the particle quasi-momenta q_1 and q_2 only, and we do not get any additional solutions apart from the class of

solutions of Eq. (2.6). Given that the expansion coefficients of the plane waves for $x < -a$ (in the region Γ_-) are known, one can reconstruct the expansion coefficients for $x > a$ (in the region Γ_+). This is a standard problem in scattering theory.

There exist two linearly independent solutions which can be taken to be symmetric and antisymmetric under the substitution $x \leftrightarrow -x$ ($\xi_1 \leftrightarrow \xi_2$):

$$\chi_+(\xi_1, \xi_2) = \exp(iKX) \begin{cases} e^{ikx} + S_+(k) e^{-ikx}; \text{ x} < -\text{a (region } \Gamma_-) \\ S_+(k) e^{ikx} + e^{-ikx}; \text{ x} > \text{a (region } \Gamma_+) \end{cases} \tag{2.13}$$

$$\chi_-(\xi_1, \xi_2) = \exp(iKX) \begin{cases} e^{ikx} - S_-(k) e^{-ikx}; \text{ x} < -\text{a (region } \Gamma_-) \\ S_-(k) e^{ikx} - e^{-ikx}; \text{ x} > \text{a (region } \Gamma_+). \end{cases} \tag{2.14}$$

The expansion coefficients in two neighboring regions of integrability are related by the scattering matrices $S_\pm(k)$. The scattering problem can be formulated on the half-axis $x \in (-\infty, 0]$ with the boundary conditions $\chi_+(X, 0)' = 0$ (symmetric case) and $\chi_+(X, 0) = 0$ (antisymmetric case), or, equivalently, on the half-axis $[0, +\infty)$. The current density vanishes for $x = 0$ for symmetric and antisymmetric wave functions. From particle number conservation it follows that the absolute values of the S-matrices are equal to unity on the real k-axis. The following properties of the S -matrix hold true in the whole complex k-plane:

$$S_\pm(k) = S_\pm(-k)^{-1},$$
$$S_\pm(k) = (S_\pm(k^*)^{-1})^*.$$

As usual, bound states are described by poles on the upper imaginary half-axis, virtual levels are described by poles on the lower imaginary half-axis. Poles in the lower half-plane of the complex k-plane correspond to resonances. For wave functions of the general form, $\chi(X, x) = C_+\chi_+(X, x) + C_-\chi_-(X, x)$, the expansion coefficients for incoming e^{ikx} and outgoing e^{-ikx} waves in the region Γ_- ($x < -a$) are connected unambiguously to the expansion coefficients in the region Γ_+ ($x > a$). The relation is expressed through the scattering matrices $S_\pm(k)$ that can be obtained by solving equation (2.10) on the interval $|\xi_2 - \xi_1| < a$. We assume that this problem is already solved and that the matrices $S_\pm(k)$ are known. In Eq. (2.13) the signs of the $S_\pm(k)$ are fixed by the convention that for free particles $S_\pm(k) = 1$ and $\chi_+(X, x) \propto \cos(kx)$, $\chi_-(X, x) \propto \sin(kx)$. Note that due to the boundary condition at $x = 0$, the value $\chi_-(x) = \exp(-iKX)\chi_-(X, x)$ can be interpreted as the radial part of the scattering wave function in three dimensions, and the value $S_-(k)$, respectively, as the S-matrix corresponding to zero angular momentum (the centrifugal potential for $l \neq 0$ cannot be included in $V(x)$ since $V(x)$ is a short range potential).

Identifying the components of the wave function $\chi(X, x) = C_+\chi_+(X, x) + C_-\chi_-(X, x)$

with the relevant terms in the expansion of Eq.(2.6), we obtain

$$\begin{aligned}
C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_s\alpha_r\ldots\alpha_N} &= C_+S_+ + C_-S_- \text{ (region } \Gamma_+, \text{wave } \mathrm{e}^{\mathrm{ikx}}),\\
C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_r\alpha_s\ldots\alpha_N} &= C_+ - C_- \qquad \text{(region } \Gamma_+, \text{wave } \mathrm{e}^{-\mathrm{ikx}}),\\
C^{\gamma_1\ldots\gamma_r\gamma_s\ldots\gamma_N}_{\alpha_1\ldots\alpha_r\alpha_s\ldots\alpha_N} &= C_+ + C_- \qquad \text{(region } \Gamma_-, \text{wave } \mathrm{e}^{\mathrm{ikx}}),\\
C^{\gamma_1\ldots\gamma_r\gamma_s\ldots\gamma_N}_{\alpha_1\ldots\alpha_s\alpha_r\ldots\alpha_N} &= C_+S_+ - C_-S_- \text{ (region } \Gamma_-, \text{wave } \mathrm{e}^{-\mathrm{ikx}})
\end{aligned} \tag{2.15}$$

where $S_\pm = S_\pm((k_{\alpha_r} - k_{\alpha_s})/2)$. This system of equations is overdetermined. Yet, it admits a consistent solution for matching the wave functions. From these equations we derive

$$C^{\gamma_1\ldots\gamma_r\gamma_s\ldots\gamma_N}_{\alpha_1\ldots\alpha_r\alpha_s\ldots\alpha_N} = \frac{S_- - S_+}{S_- + S_+} C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_r\alpha_s\ldots\alpha_N} + \frac{2}{S_- + S_+} C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_s\alpha_r\ldots\alpha_N}. \tag{2.16}$$

Combining conditions (2.8) and (2.16), we obtain

$$C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_s\alpha_r\ldots\alpha_N} = S_+((k_{\alpha_r} - k_{\alpha_s})/2) C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_r\alpha_s\ldots\alpha_N}. \tag{2.17}$$

If we make one more permutation of the indices α_s, α_r, we obtain, by virtue of $S_+((k_{\alpha_r} - k_{\alpha_s})/2)\ S_+((k_{\alpha_s} - k_{\alpha_r})/2) = 1$, the initial expression which completes this consistency check.

3 GENERALIZED BOHR-SOMMERFELD QUANTIZATION RULES

It follows from condition (2.17) that the expansion coefficients are all expressed unambiguously through $C^{1\ldots N}_{1\ldots N}$. Using Eq.(2.8), the upper indices can be ordered in sequence $(1, ..., N)$. The expansion coefficients, which can be obtained from each other by permutation of two neighboring indices, are related by Eq.(2.17). The arbitrary set of indices $(\alpha_1, ..., \alpha_N)$ can be obtained from the sequence $(1, ..., N)$ by permutations of neighboring indices. Thus, one can express $C^{1\ldots N}_{\alpha_1\ldots\alpha_N}$ through $C^{1\ldots N}_{1\ldots N}$. It remains to verify that different sequences of transpositions transforming $(1, ..., N)$ to $(\alpha_1, ..., \alpha_N)$ give the same result. Let P_1 and P_2 be two such permutations of the initial sequence $(1, ..., N)$ leading to the same final sequence $(\alpha_1, ..., \alpha_N)$. It is evident that the permutation $P_1 \times P_2^{-1}$ describes an identical (trivial) transformation. In such a sequence, each pair of indices changes places an even number of times. Since $S_+((k_{\alpha_r} - k_{\alpha_s})/2)\ S_+((k_{\alpha_s} - k_{\alpha_r})/2) = 1$, the result of the transformation $P_1 \times P_2^{-1}$ is the identity $C^{1\ldots N}_{1\ldots N} = C^{1\ldots N}_{1\ldots N}$. Denoting the result of transformation P_1 by

$$C^{1\ldots N}_{\alpha_1\ldots\alpha_N} = \alpha C^{1\ldots N}_{1\ldots N} \tag{3.1}$$

and the result of transformation P_2^{-1} by

$$C^{1...N}_{1...N} = \beta C^{1...N}_{\alpha_1...\alpha_N} \tag{3.2}$$

we obtain $\alpha\beta = 1$. Therefore, from both permutations P_1 and P_2 we obtain one and the same relation (3.1). The phase shift $\delta_+(k)$ is defined by equation $S_+(k) = \exp(2i\delta_+(k))$. With the help of eqs. (2.7), (2.8) and (2.17) we obtain the generalized Bohr-Sommerfeld quantization rule

$$k_j L + \sum_{l=1}^{N} 2\delta_+(\frac{k_j - k_l}{2}) = 2\pi n_j. \tag{3.3}$$

The sum is running over $l \neq j$, n_j are integer numbers. This equation can be interpreted in the following way. Going around the circle $[0, L]$, the particle j scatters on each particle $l \neq j$, acquiring an additional phase $2\delta_+(\frac{k_j - k_l}{2})$. When the particle j is back to the initial place, its phase turns out to be equal to the left hand side of Eq.(3.3), the additional term $k_j L$ is a result of the translation. The total phase must be an integer multiple of 2π, since the wave function is single valued.

Let us now consider the case when one virtual level exists in the complex k -plane. In this case the S-matrix

$$S_+(k) = (k - ik_0)/(k + ik_0), \tag{3.4}$$

where $k_0 = m\alpha > 0$, corresponds to the delta-function potential $V(x) = \alpha\delta(x)$ (problem 2.47 in ref. [20]). Respectively, $\delta_+(-\infty) = 0$, $\delta_+(+\infty) = 2\pi$, $\delta_+(0) = \pi$, so that $S_+(0) = -1$. It is seen from eqs.(2.13) that the symmetric wave function vanishes for $k = 0$ and $S_+(0) = -1$ and the particle quasi-momenta in the set $(k_1, k_2, ..., k_N)$ must be all distinct. Therefore, one obtains a constraint which resembles the Pauli principle even though we have started with a system of bosons. In the lowest energy state, the particle quasi-momenta $(k_1, k_2, ..., k_N)$ occupy the Fermi sphere. The equality $S_+(0) = -1$, along with the exclusion principle, apparently, is valid for any odd number of virtual levels.

For an even (zero) number of virtual levels one has $S_+(0) = 1$, and some quasi-momenta in the set $(k_1, k_2, ..., k_N)$ can coincide. However, for $q_1 = q_2$, $|x| > a$, Eq. (2.10) has the general solution $\chi_+(X, x) = (a + bx)\exp(iKX)$ with $b \neq 0$. For this reason it is impossible to satisfy the periodic boundary condition. The states with $b \neq 0$ correspond to zero-energy scattering states. The coefficient b vanishes for a special class of potentials having a representation of the form

$$V(x) = \frac{1}{m}\chi''(x)/\chi(x) \tag{3.5}$$

with the wave function satisfying the condition $\chi'(0) = \chi'(a) = 0$. From $V(a) = 0$ it also follows that $\chi''(a) = 0$. Solutions of the Schrödinger equation with a limited

asymptotic behavior at infinity (i.e. $b = 0$) exist when a discrete level in the potential appears (problem 2.18 in ref. [20]). Therefore, it is clear that the case $b = 0$ is an exceptional one. Solutions of such a kind occur when we pass to a system in which bound states are formed.

We thus consider repulsive potentials without discrete states in the energy spectrum. For such potentials all quasi-momenta in the set $(k_1, k_2, ..., k_N)$ must be distinct.

4 THERMODYNAMIC LIMIT

In the thermodynamic limit $N \to \infty$, $L \to \infty$, and $\rho = N/L$ a fixed value, the particles in the ground state occupy continuously the Fermi sphere. The sum over the particle quasi-momenta can be approximated by an integral over k with a weight function $f(k)$

$$\sum \to \int \frac{Ldk}{2\pi} f(k),$$

where $Lf(k)dk/(2\pi)$ is the number of states in the interval dk. The distribution function $f(k)$ can be found from equation

$$f(k) = 1 + \int_{-p_F}^{p_F} \frac{dk'}{2\pi} f(k') \delta'_+ (\frac{k - k'}{2}) \tag{4.1}$$

where $\delta'_+(\frac{k-k'}{2})$ is the derivative of the scattering phase shift with respect to the argument. This equation is obtained by rewriting Eq.(3.3) in the thermodynamic limit for the phase difference of the particles $j+1$ and j taking into account that $n_{j+1} - n_j = 1$. Eq.(4.1) is a generalization of Eq.(3.12) in ref. [3] to arbitrary singular potentials.

We consider the problem of finding the spectrum of elementary excitations. Let us consider two sets of particle quasi-momenta $(k_1, k_2, ..., k_N)$ and $(k'_1, k'_2, ..., k'_N)$. In the first set the k's occupy continuously the Fermi sphere. The second set of k's is obtained from the first one by removing a particle from the Fermi surface with the quasi-momentum $k_N = p_F$ and giving it an arbitrary quasi-momentum $q = k'_N > p_F$ or $q = k'_N < -p_F$. Because of the interaction, the particle quasi-momenta inside the Fermi sphere receive a shift $k'_j - k_j = \Delta(k_j)/L$. Taking the difference between the two relations

$$k_j L + \sum_{l=1}^{N-1} 2\delta_+ (\frac{k_j - k_l}{2}) + 2\delta_+ (\frac{k_j - p_F}{2}) = 2\pi n_j,$$

$$k'_j L + \sum_{l=1}^{N-1} 2\delta_+ (\frac{k'_j - k'_l}{2}) + 2\delta_+ (\frac{k'_j - p_F}{2}) = 2\pi n_j,$$

we obtain with the help of Eq.(4.1),

$$\Delta(k)f(k) = -2\delta_+(\frac{k-q}{2}) + 2\delta_+(\frac{k-p_F}{2}) + \int_{-p_F}^{p_F} \frac{dk'}{2\pi}\Delta(k')f(k')\delta'_+(\frac{k-k'}{2}). \quad (4.2)$$

The total momentum of the system equals

$$P = q - p_F + \int_{-p_F}^{p_F} \frac{dk'}{2\pi}\Delta(k')f(k'). \quad (4.3)$$

The first term is the quasi-momentum of the excited particle. After removing the particle with quasi-momentum p_F the momentum of the Fermi sphere changes by an amount $-p_F$. The last term represents the change of the total momentum of the particles inside the Fermi sphere. The energy of the system equals

$$\epsilon = \frac{q^2}{2m} - \frac{p_F^2}{2m} + \int_{-p_F}^{p_F} \frac{dk'}{2\pi}\frac{k'}{m}\Delta(k')f(k'). \quad (4.4)$$

With $q \longrightarrow p_F$ the momentum P and the energy ϵ of the excited state vanish. In order to find the dispersion law for elementary excitations, $\epsilon(P)$, it is necessary to express q in terms of P with the help of Eq.(4.3) and substitute the resulting expression into Eq.(4.4). Note that the integral equations (4.1) and (4.2) have the same kernel

$$R(k,k') = (2\pi)\delta(k-k') - \delta'_+(\frac{k-k'}{2}) \quad (4.5)$$

and that $R(k,k')$ is symmetric. Solutions of these equations can be represented in the form

$$f(k) = \int_{-p_F}^{p_F} \frac{dk'}{2\pi}R^{-1}(k,k'), \quad (4.6)$$

$$\Delta(k)f(k) = \int_{-p_F}^{p_F} \frac{dk'}{2\pi}R^{-1}(k,k')(-2\delta_+(\frac{k'-q}{2}) + 2\delta_+(\frac{k'-p_F}{2})). \quad (4.7)$$

Integrating Eq.(4.7) over k and using the symmetry under permutation of the arguments of $R^{-1}(k,k')$, we can rewrite Eq.(4.3) in the form

$$P = q - p_F + \int_{-p_F}^{p_F} \frac{dk'}{2\pi}f(k')(-2\delta_+(\frac{k'-q}{2}) + 2\delta_+(\frac{k'-p_F}{2})). \quad (4.8)$$

In lowest order of the difference $q - p_F$

$$P = (q-p_F)f(p_F), \quad (4.9)$$

$$\epsilon = (q-p_F)\int_{-p_F}^{p_F} \frac{dk'}{2\pi}\frac{k'}{m}R^{-1}(k',p_F). \quad (4.10)$$

In a similar way one can treat excitations which transfer a particle to the opposite side of the Fermi surface, that is for $k_1 = -p_F$ and $q = k'_1 > p_F$ or $q = k'_1 < -p_F$. These excitations are described by the same formulae, if the substitution $p_F \longrightarrow -p_F$ is made.

Other kinds of elementary excitations of "hole" type (non-Bogoliubov excitations) are obtained by taking away a particle with quasi-momentum q inside of the Fermi sphere and placing it on the Fermi surface. Let us find the spectrum of such excitations. Consider two sets of the particle quasi-momenta $(k_1, k_2, ..., k_N)$ and $(k'_1, k'_2, ..., k'_N)$. In the first set the quasi-momenta occupy continuously the Fermi sphere, while the second set is obtained from the first one by removing a particle with a quasi-momentum $k_m = q$ from the Fermi sphere and placing it on the Fermi surface at $k'_m = p_F$. By comparison of the relations

$$k_j L + \sum_{l=1}^{N-1} 2\delta_+(\frac{k_j - k_l}{2}) + 2\delta_+(\frac{k_j - q}{2}) = 2n_j,$$

$$k'_j L + \sum_{l=1}^{N-1} 2\delta_+(\frac{k'_j - k'_l}{2}) + 2\delta_+(\frac{k'_j - p_F}{2}) = 2n_j$$

with the corresponding relations for "particle" type excitations, discussed before, we see that the corresponding equations can be obtained from the ones already discussed by the replacement $p_F \to q$. In the above formulae the terms $l = m$ are excluded from the summation.

In the symmetric case, when the particle with quasi-momentum $k_m = q$ is placed on the Fermi surface at $k'_m = -p_F$, it is necessary to make the replacement $p_F \to -p_F$.

5 EXACTLY SOLVABLE MODELS FOR FERMI-SYSTEMS

In full analogy to the Bose case, the problem of constructing the wave function of the Fermi-system can be solved when the spins of the fermions are all lined up in one direction. In such a case, the spin part of the wave function is symmetric, and the coordinate part is totally antisymmetric. Let us find how eqs.(2.7), (2.8), (2.17) and (3.3) should be modified.

The periodic boundary condition (2.7), apparently, remains unaltered.

Under permutations of the coordinates, the wave function is antisymmetric. The symmetry conditions yield the expansion coefficients satisfying the condition:

$$C^{\gamma_1 \dots \gamma_s \dots \gamma_r \dots \gamma_N}_{\alpha_1 \dots \alpha_s \dots \alpha_r \dots \alpha_N} = -C^{\gamma_1 \dots \gamma_r \dots \gamma_s \dots \gamma_N}_{\alpha_1 \dots \alpha_s \dots \alpha_r \dots \alpha_N}. \quad (5.1)$$

In comparison to the Bose case, the right hand side acquires a minus sign.

The matching conditions of the wave function at the boundaries of the different regions of integrability are derived as for bosons. Relations (2.16) are valid.

Combining eqs.(2.16) and (5.1), we obtain

$$C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_s\alpha_r\ldots\alpha_N} = S_-((k_{\alpha_r} - k_{\alpha_s})/2)C^{\gamma_1\ldots\gamma_s\gamma_r\ldots\gamma_N}_{\alpha_1\ldots\alpha_r\alpha_s\ldots\alpha_N}. \tag{5.2}$$

The expansion coefficients are related to each other through the matrix $S_-(k)$ that describes scattering on the positive half-axis with the boundary condition $\chi(0) = 0$.

The generalized Bohr-Sommerfeld quantization rules have a form analogous to Eq.(3.3)

$$k_jL + \sum_{l=1}^{N} 2\delta_-(\frac{k_j - k_l}{2}) = 2\pi n_j. \tag{5.3}$$

where the phase shift is defined by $S_-(k) = \exp(2i\delta_-(k))$. The summation is performed over $l \neq j$, n_j are integer numbers.

Therefore, there is a close analogy between the the behavior of Bose- and Fermi-systems with parallel spins. This analogy also exists in the thermodynamic limit. The results of Sect.4 are valid for Fermi-systems after the replacement $\delta_+(k) \leftrightarrow \delta_-(k)$.

In the limit $k \to 0$, the scattering phase shifts $\delta_+(k)$ and $\delta_-(k)$ have a different behavior. For smooth potentials, the continuity conditions for the logarithmic derivatives at $x = a$ in the symmetric and antisymmetric cases have the form

$$k \tan(ka + \delta_+(k)) = \kappa_+, \tag{5.4}$$

$$k \cot(ka + \delta_-(k)) = \kappa_-, \tag{5.5}$$

k_+ and k_- do not depend on the momentum k if $k << \kappa_+, \kappa_-$. We conclude that at small k the scattering phase shifts have the form

$$\delta_+(k) = -ka + \arctan(\kappa_+/k), \tag{5.6}$$

$$\delta_-(k) = -ka + \mathrm{arccot}(\kappa_-/k). \tag{5.7}$$

In the limit $a \to 0$ and for arbitrary small but finite values of κ_+ and κ_-, we obtain

$$\delta_+(k) \to \pi\theta(k), \tag{5.8}$$

$$\delta_-(k) \to 0. \tag{5.9}$$

In the weak coupling regime (the value of the κ_+ is small) the quantization conditions (2.17) for the symmetric case remain non-trivial, since the derivative of $\delta_+(k)$ is proportional to a delta-function. The integral kernel $R(k, k')$ defined by Eq.(4.5) is determined by the difference of the delta-function and the derivative of $\delta_+(k)$. For

Bose-systems with the potential $V(x) = \alpha\delta(x)$, it is impossible to reproduce the result of perturbation theory analytically beyond first order of α [4], because of the complicated character of the kernel $R(k,k')$. In general it is, however, not difficult to perform a comparison with perturbation theory numerically. The non-trivial character of the weak coupling regime can be related to the non-analyticity in the coupling constant (in λ if we write the potential as $\lambda V(x)$), that should be present because of the instability of the system of bosons when the sign of the potential is changed.

According to Eq.(5.9) the phase $\delta_-(k)$ and its derivative should be set equal to zero. After that, the quantization conditions (5.3) take a very simple form $k_j L = 2\pi n_j$. As a result, we deal with a Fermi-gas of non-interacting particles. The weak coupling regime is therefore trivial for fermions.

6 BOSE-EINSTEIN CONDENSATION OF TWO-FERMION RESONANCES

In this sect. we consider a problem of considerable physical interest. As in sect. 5 consider a one-dimensional system of fermions interacting via a finite ranged potential $V(x)$. Suppose there exists in the fermion-fermion channel a resonance with momentum $k_0 = k_1 - ik_2$, where k_1 and k_2 are real, positive numbers. One expects that after increasing the Fermi momentum k_F above the value of the resonance k_1, the creation of such resonances will be energetically favored by the system. The resonances can be treated as composite Bose particles. If their interaction is small, they are accumulated in the ground state like real bosons with zero total momentum. The Bose-Einstein condensation reveals itself by an increase of the distribution function $f(k)$ of the particle quasi-momenta near the Fermi surface, since the resonances are at rest and the fermion momenta are concentrated in the vicinity of $k \approx k_1 \approx k_F$. The effect is well pronounced for a small width and disappears with increasing width of the resonance.

Near the resonance, the S-matrix can be parametrized in the Breit-Wigner form

$$S_-(k) = \frac{(k+k_0)(k-k_0^*)}{(k+k_0^*)(k-k_0)}, \tag{6.1}$$

An S-matrix of such a type corresponds not only to finite range potentials, but also to zero-range singular potentials.

Let us consider a potential of the form

$$V(x) = -V_0\theta(a-x) + \alpha\delta(x-a), \tag{6.2}$$

where V_0 represents a positive value. We wish to find a solution of the scattering problem on the half-axis $[0, +\infty)$. The wave function takes the form

$$\chi(x) = \begin{cases} \sin(Kx); & 0 < x < a, \\ C\sin(kx + \delta\_(k)); & a < x. \end{cases} \tag{6.3}$$

Here, $K = \sqrt{k^2 + V_0}$ (in "atomic units" $m = \hbar = 1$). The scattering phase is determined by the condition

$$k\cot(ka + \delta_-(k)) = K\cot(Ka) + \alpha. \tag{6.4}$$

The S-matrix takes the form

$$S_-(k) = \exp(-2ika)\frac{K\cot(Ka) + \alpha + ik}{K\cot(Ka) + \alpha - ik}. \tag{6.5}$$

The S-matrix poles can be found from the equation

$$\tan(Ka) = \frac{K}{ik - \alpha}. \tag{6.6}$$

In the limit $\alpha \to \infty$, the S-matrix poles cross the real axis at $Ka = 0$ ($mod\ \pi$). The lowest level is located at $Ka = \pi$. We may keep the lowest resonance momentum $k_1 = \sqrt{\pi^2/a^2 - V_0}$ fixed and pass to the limit $V_0 \to \infty$ and $a \to 0$. The zero-width resonance occurs then from a solution to the Schrödinger equation with the zero-range singular potential (6.2). The potential is determined by the limiting procedure: $V_0 \to \infty$, $a \to 0$, $k_1 = \sqrt{\pi^2/a^2 - V_0} = constant$. It generates on the real k-axis exactly two poles (one zero-width resonance) in the S -matrix. The other poles are moved to infinity.

Let now α be finite, but large, and V_0 receives a correction $\Delta V_0 << V_0$. The resonance acquires, first of all, a finite width. We require the ΔV_0 be such that the real part of the resonance momentum be equal to k_1. The S-matrix pole is located at $k_0 = k_1 - ik_2$.

Now, we parametrize $\alpha = 1/(\xi x^3)$ and $a = x^2$, so that

$$V_0/\pi^2 = \frac{1}{x^4} - k_1^2. \tag{6.7}$$

In the limit $x \to 0$, Eq.(6.6) gives

$$\Delta V_0/\pi^2 = -\frac{2\xi}{x^3} + \frac{3\xi^2}{x^2} - \frac{2\xi^3}{3x}(6 - \pi^2 - 3x^4k_1^2) + \frac{\xi^4}{3}(10 - 13\pi^2 - 27x^4k_1^2) + \ldots\ , \tag{6.8}$$

$$k_2/\pi^2 = \xi^2 - 3x\xi^3 + (6 - \pi^2 - x^4k_1^2)x^2\xi^4 + \ldots\ . \tag{6.9}$$

As a result, we obtain a zero-range singular potential for which the S -matrix in the complex k-plane has two poles only, which can be identified with a resonance. The resonance width is proportional to ξ^2. For narrow resonances $\xi << 1$.

In Fig.1, we show results for the distribution function $f(k)$ for different values of the Fermi momentum k_F from 0.64 to 1.6 with a step 0.12 in a system where the two-fermion interaction is described by a S-matrix of the form (6.1) with $k_1 = 1$, $k_2 = 0.05$ (small width). The distribution function $f(k)$ increases near the Fermi surface where the production of the resonances becomes energetically favorable. This effect can be interpreted in terms of a Bose-Einstein condensation of the resonances whose wave function are concentrated at $k \approx k_1 \approx k_F$. The value k_2 measures the spread of the fermion momenta in the resonances. When the Fermi momentum is increased beyond the resonance momentum k_1 the distribution function shows a plateau centered at $k \approx k_1$. If all resonances are in the Bose-Einstein condensate, the size of the plateau would be of order k_2. However, its size increases with the Fermi momentum. This can be interpreted as follows: It is known that in systems of interacting bosons there exists a fraction of bosons with nonvanishing velocities which are not in the condensate and which have a nontrivial momentum distribution [21]. These bosons contribute to the pressure. Their fraction increases with the total density of the bosons.

The appearance of resonances in Fermi systems yields a softer equation of state. This means that the pressure p increases slower with the density n . The effect is shown in Fig. 2. The sudden rise of $p = p(n)$ appears at $k_F = k_1$ where the particle density equals $n = 0.36$.

For a broad resonance, the derivative of the scattering phase $\delta'(\frac{k-k'}{2})$ entering Eq.(4.2) is small, the distribution function $f(k)$ is close to unity, and therfore the effect of the resonance on the equation of state is small, too.

7 SELF-GRAVITATING OBJECTS

Bose-Einstein condensation of narrow two-fermion resonances may drastically change the properties of fermion matter, producing physically interesting phenomena, for example, the instability of neutron stars. Before, studying this effect in more detail let us qualitatively describe the expected scenario.

Narrow two-fermion resonances can be treated as Bose particles. If the central density of a neutron star exceeds a critical value, creation of these bosons with subsequent formation of a Bose-Einstein condensate becomes energetically favorable. The critical density for a Bose-Einstein condensate formation is determined by the mass of the resonances. In the ideal Bose-gas approximation, the chemical potential of the fermions is frozen at $\mu = m_D/2$, where m_D is the dibaryon mass. The dibaryons are at rest. Therefore, they do not collide with the boundary and do not contribute to the pressure. The pressure is determined by the fermions only. The

number of fermions remains constant with increasing density, since the radius of the Fermi sphere is frozen, whereas the number of the resonances increases linearly. The incompressibility of matter vanishes. In this way, we give a qualitative explanation for the observed growth of the distribution function $f(k)$ near the Fermi surface in Fig. 1 as well as the rapid change of $p = p(n)$ in Fig. 2.

Suppose that the short-range potential between fermions is such that a narrow two-fermion (dibaryon) resonance exists and that a Bose-Einstein condensate of these resonances is formed in the interior of a neutron star (in three dimensions) for $r < r_1$. In the inner region, due to the formation of the resonances, the pressure remains constant, i.e. $\nabla p = 0$. Gauss's law implies

$$\int d\mathbf{S} \cdot \nabla\phi(r) = 4\pi G M(r)$$

where $M(r)$ is mass of the substance inside of a sphere of the radius r and G the gravitational constant. We conclude that $\nabla\phi(r) \neq 0$. The Euler equation

$$\rho\frac{\partial \mathbf{v}}{\partial t} + \rho(\mathbf{v}\cdot\nabla)\mathbf{v} = -\nabla p - \rho\nabla\phi(r) \tag{7.1}$$

shows that in the static case ($\mathbf{v} \equiv 0$) the gradient of the pressure p is balanced by the gradient of the gravitational potential $\phi(r)$. However, if $\nabla p = 0$, static solutions are impossible and neutron stars are gravitationally unstable.

Let us study this effect in a one-dimensional model of the neutron star, where all quantities can be calculated exactly, in order to test if some instability occurs due to the formation of the two-fermion resonances. We restrict ourselves to nonrelativistic Newtonian gravity. The static stable configurations of neutron stars are described by Euler's equation with a vanishing left hand side

$$-\frac{dp(x)}{dx} - \rho(x)\frac{d\phi(x)}{dx} = 0 \tag{7.2}$$

where $p(x)$ is the pressure, $\rho(x) = mn(x)$ the mass density, $n(x)$ the number density, and $\phi(x)$ the gravitational potential. The center of the neutron star is placed at the origin of the coordinates. Gauss's law gives

$$\frac{d^2\phi(x)}{dx^2} = 4\pi G\rho(x). \tag{7.3}$$

Note that in contrast to classical electrodynamics the right hand side of Eq.(7.3) has a positive sign.

We integrate Eq.(7.3) from $-x$ to $+x$. Using the symmetry of the potential $\phi(x)$ under the reflection $x \leftrightarrow -x$ ($p(x)$ and $\rho(x)$ are also symmetric functions) and the antisymmetry of the first derivative of $\phi(x)$, one gets

$$\frac{d\phi(x)}{dx} = 2\pi G M(x), \tag{7.4}$$

where

$$M(x) = \int_{-x}^{x} dx\rho(x) = 2\int_{0}^{x} dx\rho(x). \tag{7.5}$$

Substituting Eq.(7.4) into Eq.(7.2), we obtain

$$\frac{dp}{dM} + \pi G M = 0, \tag{7.6}$$

and finally

$$M(x) = \sqrt{\frac{2}{\pi G}(p(0) - p(x))}. \tag{7.7}$$

At the surface $p(x_s) = 0$, and the total mass of the neutron star $M_s = M(x_s)$ is expressed unambiguously through the central pressure $p(0)$. The neutron star radius x_s can be obtained from Eq.(7.7). It is sufficient to take the derivative of Eq.(7.7) with respect to x, divide both sides by $2\rho(x)$, and integrate the result over x. In this way one gets

$$x_s = \frac{1}{\sqrt{2\pi G}} \int_{0}^{p(0)} \frac{dp}{\rho(p)\sqrt{p(0) - p}} \tag{7.8}$$

Assuming that the equation of state $n = n(p)$ is known, eqs.(7.7) and (7.8) allow to determine the neutron star radius x_s as a function of the neutron star mass M_s, or, equivalently, the total mass M_s as a function of the central pressure $p(0)$. The criterion for the gravitational stability of stars has the form (see ref. [22], Eq.(10.1.4p))

$$\frac{\partial x(M)}{\partial M_s} < 0, \tag{7.9}$$

where $x(M)$ is a coordinate, such that inside of the interval $[-x, x]$ the matter of mass M is contained:

$$x(M) = \int_{0}^{M} \frac{dM'}{2\rho}. \tag{7.10}$$

Taking the derivative with respect to M_s and using Eq. (7.7) to express the pressure in terms of the mass, we obtain

$$\frac{\partial x(M)}{\partial M_s} = -\frac{\pi G M_s}{2} \int_{0}^{M} \frac{dM'}{\rho^2}\frac{d\rho}{dp} < 0. \tag{7.11}$$

In the non-degenerate case $dp/d\rho = a_s^2 > 0$ (a_s is the velocity of sound), the derivative $\partial x(M)/\partial M_s$ is negative definite and one-dimensional neutron stars are in general gravitationally stable.

In three-dimensional space this is not necessarily the case. In order to investigate the reason for this difference, let us express the sum of the internal energy and the gravitational binding energy

$$E = \int \epsilon dV + \mathrm{sign}(2 - D)G \int r^{2-D} M dM \tag{7.12}$$

in terms of the average density $\bar{\rho} \sim M/r^D$ where D is the dimension of the space. For an equation of state of the form $\varepsilon = A\rho^\gamma$ we get

$$E = A\bar{\rho}^{\gamma-1}M + \mathrm{sign}(2-D)BM^{(2+D)/D}\bar{\rho}^{(D-2)/D} \tag{7.13}$$

where B is a fixed positive constant.

In the one-dimensional case $D = 1$, a minimum energy E as a function of $\bar{\rho}$ is obtained for $\gamma > 1$. In the ideal gas approximation for the Bose-Einstein condensate $\gamma = 1$, $\rho \sim m_D n$, where n is the number density, and so there is no stability. The maximum pressure determines the maximum mass of the neutron star

$$M_{\max} = \sqrt{\frac{2}{\pi G} p_{\max}(0)}. \tag{7.14}$$

However, if one goes beyond the ideal gas approximation, after the creation of the resonances the pressure still increases slowly with the density due to the interactions between the fermions and bosons as shown in Fig. 2. In the one-dimensional problem, even an arbitrarily slow growth of the pressure ($\gamma = 1 + 0.0...01$) stabilizes the neutron star.

In the three-dimensional case, the minimum of E exists for $\gamma > 4/3$, and for the stabilization of the neutron star, an arbitrarily slow growth of pressure is insufficient. In order to make the neutron star stable, it is necessary to increase the average value of γ above 4/3 (see [22]). Consequently, in the presence of a dibaryon condensate a considerably stiffer equation of state is required to prevent the neutron star from collapsing.

8 CONCLUSION AND DISCUSSIONS

We have shown that the properties of one-dimensional Bose- and some Fermi-systems can be determined exactly without recourse to perturbation theory for a wide class of singular potentials. The solutions are expressible through two-particle scattering phase shifts. The Fermi character of the energy spectrum in one-dimensional Bose-systems is not specific to potentials of delta-function type. In any singular potential there are hole-type non-Bogoliubov branches of elementary excitations. We have derived the dispersion laws for these excitations.

The one-dimensional Fermi system can be analyzed exactly if the fermion spins are lined up in one direction. The quantization rules have a similar form for Bose- and Fermi-systems.

An exact analysis of Fermi-systems with a resonance in the two-fermion channel is given. In the model considered, the Pauli principle and the composite nature of

the resonances are taken into account at the outset. We have observed an increase of the distribution function $f(k)$ of fermions over the quasi-momenta k in the vicinity of the Fermi surface when the density is close to the critical density of resonance formation. We could interpret this behavior in terms of a Bose-Einstein condensation of two-fermion resonances. The formation of the resonances is accompanied by a softening of the equation of state. In the real (three-dimensional) world softening of the equation of state of nuclear matter caused by dibaryon resonances can produce instability of neutron stars. In a one-dimensional model of self-gravitating objects this effect does not exist.

The existence of a dibaryon resonance will lead at higher densities to the occurrence of a new state of nuclear matter. The dibaryons behave like bosons and can form a Bose-Einstein condensate in nuclear matter. With increasing baryon density, the pressure should only slightly increase with the density. In a perturbative picture (which was not used here), the dibaryon condensation should occur at a Fermi energy ε_F determined by the condition

$$2(m_N + \varepsilon_F) \geq m_D$$

where m_N and m_D are the nucleon and dibaryon masses.

It will be interesting to search for such a dibaryon condensate in heavy-ion collisions. In the center-of-mass frame of the condensate a large fraction of dibaryons has zero velocities. When the density decreases, dibaryons in the condensate decay into nucleons. If a channel $D \rightarrow NN$ exists, experimentalists will observe monochromatic nucleons with the energy $m_D/2$. Since the rest frame of the condensate is *a priori* unknown, it is necessary to look for a boost transformation along the beam momentum into a coordinate system in which a statistically significant excess of monochromatic nucleons exists. An excess of such events can be considered as a possible signature for the formation of a dibaryon condensate.

It is also interesting to check astrophysical data for the presence of a dibaryon condensate in the interiors of massive neutron stars as well as for signatures of instability of neutron stars caused by dibaryons.

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FIGURE CAPTIONS

Fig.1 The distribution function $f(k)$ versus the quasi-momentum $k < k_F$ for 8 different Fermi momenta k_F ranging from 0.64 to 1.6 with a step 0.12 (in units $m = \hbar = 1$). The two-body S-matrix has a pole corresponding to a resonance at $k = k_1 - ik_2$ with $k_1 = 1$ and $k_2 = 0.05$ (i.e. small width). The distribution function $f(k)$ increases near the Fermi surface when k_F approaches k_1. This effect can be interpreted in terms of a Bose-Einstein condensation of the resonances, since the wave function of the resonance is concentrated at $k \approx k_1$.

Fig.2 Pressure p versus fermion number density n. The change of the slope occurs at $n = 0.36$ when $k_F = k_1$ (in units $m = \hbar = 1$). This is the point at which the two-fermion condensation starts. It results in a softening of the equation of state. In the ideal gas approximation, the pressure does not increase with the density after the condensation has set in. If interactions are included, the pressure is slowly increasing even after the condensation.